

Notes on undecidability of Medvedev's logic

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Abstract

We prove that Medvedev's intermediate logic is undecidable and is not recursively enumerable. The proof starts from the bounded canonical formulas of Zakharyashev in their algebraic presentation. For a finite subdirectly irreducible Heyting algebra A , with finite rooted dual poset P , failure of the canonical formula is reduced to the existence of an onto cofinal partial Esakia morphism from an entire Medvedev frame to P satisfying the selected closed-domain conditions. We then encode the periodic Wang domino problem into such finite partial morphisms. Four kinds of maximal vertices encode horizontal and vertical phases; guard chains force directed cycles; binary points encode successor and tile labels; and additional shield points distinguish incident from nonincident constraints. The converse direction is obtained from a periodic tiling by a calibrated support construction on a finite simplex. This gives a computable reduction from the periodic domino problem to non-theoremhood in Medvedev's logic. We then prove a raw-frame version of the canonical criterion and transfer the construction to Skvortsov's infinite Medvedev frame. This shows that Skvortsov's recursively axiomatizable logic is undecidable and is properly contained in Medvedev's logic. We separately analyze the Jankov-generated core Y_{ML} . Its recursive enumerability is equivalent to decidability of finite Medvedev-frame recognition; neither this recognition problem nor the proposed identification $Y_{\text{ML}} = \text{Skv}$ is resolved by the selected-join construction. Finally, we express the finite reduction as selected cover-exact face filling in standard simplices and identify ML with the universally quantified propositional theory of the topos of simplicial sets.

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1 Methodology and Context

We here explain the methodology for the present working note. We explain as it stands the meaning of the two roles appearing: the “prompter” is the person who employed the Chat GPT tool to obtain the desired results, but whose expertise does not necessarily align completely with the problem - in this case, despite working in superintuitionistic logics, Almeida has never worked in issues of decidability of logics. His expertise is limited to knowing enough of the field and of the meaning of the relevant concepts to guide the model slightly towards the solution. The key motivation for prompting this question was in part the desire that his friend Soren Knudstorp not go through what he recently went through, namely, finding out that someone else had used a model to find a solution, and instead he be given the option of what to do with this proof and turn it into the best version it could possibly be.

On the other hand, Knudstorp's expertise, insight and alignment with the key issues of the problem are invaluable for the eventual solution, and so he has been deemed by Almeida as proof caretaker, should he accept the task of carefully working through this proof.

The prompt guidelines given were as follows: the model was instructed to employ Bezhanishvili and Zacharyashev's canonical formula axiomatization of superintuitionistic logics. After clarifying the precise criterion, the model was furthermore asked to obtain a simplification of this axiomatization by employing structural completeness and the disjunction property, structural hypotheses which might (Knudstorp will determine) have simplified the search space.

Subsequently the model was asked to execute a parallel search in several areas: graph combinatorics, CSP, finite topology combinatorics. It was tasked with creating several teams working towards a positive solution, as well as one team working towards a negative one, where it was specifically asked to employ the methods and ideas developed by Knudstorp in his recent work on undecidability of several logics. After two iterations the model produced a proof. The present note contains the summarized contents of that argument.

2 Introduction

For $n \geq 1$, the n -th Medvedev frame is

$$M_n = (\mathcal{P}^*([n]), \supseteq), \quad \mathcal{P}^*([n]) = \mathcal{P}([n]) \setminus \{\emptyset\}.$$

Thus $S \leq T$ in M_n precisely when $S \supseteq T$. Its least point is $[n]$, its maximal points are the singletons, and

$$\uparrow S = \{T : \emptyset \neq T \subseteq S\}.$$

Medvedev's logic is

$$\text{ML} = \bigcap_{n \geq 1} \text{Log}(M_n).$$

Equivalently, if $B_n = \text{Up}(M_n)$ denotes the Heyting algebra of upsets of M_n , then a formula belongs to ML exactly when it is valid in every B_n .

Medvedev's logic has the disjunction property and is structurally complete [20]. It is also known not to be finitely axiomatizable [16]. Recent accounts have continued to list its decidability, equivalently its recursive axiomatizability, as an open problem [8, 7]. The purpose of this note is to settle this problem negatively.

Theorem 2.1 (Main theorem). *The set of formulas valid in every Medvedev frame is co-recursively enumerable but not recursively enumerable. Consequently, Medvedev's logic is undecidable and has no recursively enumerable axiomatization.*

We obtain a second formulation in terms of the canonical axiomatization.

Theorem 2.2 (Canonical-pattern problem). *There is no algorithm which, given a finite rooted poset P and a finite set $D \subseteq \text{Up}(P)^2$, decides whether the bounded canonical formula $\rho(\text{Up}(P), D)$ belongs to ML. This set of positive canonical patterns is co-recursively enumerable but not recursively enumerable.*

The proof uses the doubly periodic domino problem. The key point is not merely to encode local tile compatibility. An onto partial morphism may use supports of arbitrary cardinality, and an arbitrary relation support can interact with a tile support without sharing the intended vertex. The guard and shield points introduced below rule out precisely these two evasions.

3 Canonical formulas and their dual semantics

3.1 Finite Heyting algebras and rooted duals

For a finite poset P , the set $\text{Up}(P)$ of upsets is a Heyting algebra. Meet and join are intersection and union, while

$$U \rightarrow V = P \setminus \downarrow (U \setminus V).$$

A finite Heyting algebra A is subdirectly irreducible exactly when $A \setminus \{1\}$ has a largest element s , called its opremum. Under finite Esakia duality, finite subdirectly irreducible Heyting algebras correspond to finite rooted posets, where *rooted* means having a least point. Indeed, if P has root r , then $P \setminus \{r\}$ is the largest proper upset of P .

3.2 The bounded canonical formula

Let A be a finite subdirectly irreducible Heyting algebra with opremum s , and let $D \subseteq A^2$ be arbitrary. Introduce a variable p_a for each $a \in A$, and put

$$\begin{aligned} \Delta(A, D) = & \bigwedge_{a,b \in A} (p_{a \wedge b} \leftrightarrow (p_a \wedge p_b)) \\ & \wedge \bigwedge_{a,b \in A} (p_{a \rightarrow b} \leftrightarrow (p_a \rightarrow p_b)) \\ & \wedge \bigwedge_{a \in A} (p_{\neg a} \leftrightarrow \neg p_a) \\ & \wedge \bigwedge_{(a,b) \in D} (p_{a \vee b} \leftrightarrow (p_a \vee p_b)). \end{aligned}$$

The bounded canonical formula associated with (A, D) is

$$\rho(A, D) = \Delta(A, D) \rightarrow p_s.$$

This is the formula denoted $\alpha(A, D)$ in [4, Definition 4.1]. It is equivalent to Zakharyashev's original relational canonical formula [4, Remark 5.13]. The adjective “bounded” is important: the negation diagram forces preservation of the bottom element and corresponds dually to cofinality. Some older conventions write an additional $\perp$ parameter for this version.

The set D is not required to satisfy a closure axiom. It records exactly the pairs whose joins must be preserved by the diagram map.

3.3 Partial Esakia morphisms

We recall the finite form of the generalized Esakia duality used for these formulas. Let X, Y be finite posets and let $f : X \rightarrow Y$ be a partial map. We write $f[\uparrow x]$ for the images of domain points in $\uparrow x$.

Definition 3.1. The map $f : X \rightarrow Y$ is a *partial Esakia morphism* when:

- (PE1) if $x, z \in \text{dom}(f)$ and $x \leq z$, then $f(x) \leq f(z)$;
- (PE2) if $x \in \text{dom}(f)$ and $f(x) \leq y$, then some $z \in \text{dom}(f)$ satisfies $x \leq z$ and $f(z) = y$;
- (PE3) $x \in \text{dom}(f)$ exactly when $f[\uparrow x] = \uparrow y$ for some $y \in Y$;
- (PE4) $f[\uparrow x]$ is closed for every $x \in X$;
- (PE5) for each clopen upset $U \subseteq Y$, the set $X \setminus \downarrow f^{-1}(Y \setminus U)$ is a clopen upset of X .

It is *cofinal* when, for every $x \in X$, there is a $z \in \text{dom}(f)$ with $x \leq z$.

Finite Esakia spaces have the discrete topology. Thus the topological parts of (PE4) and (PE5) reduce to finiteness; the set in (PE5) is an upset because it is the complement of a downset.

Let P be the finite rooted dual of A , and write $\zeta(a) \subseteq P$ for the upset corresponding to $a \in A$. For upsets $U, V \subseteq P$, define

$$\mathcal{D}_{U,V} = \left\{ C \in \text{Ant}(P) : \begin{array}{l} C \subseteq U \cup V, \\ C \cap (U \setminus V) \neq \emptyset, \\ C \cap (V \setminus U) \neq \emptyset \end{array} \right\},$$

and put

$$\mathcal{D}_D = \bigcup_{(a,b) \in D} \mathcal{D}_{\zeta(a), \zeta(b)}.$$

Definition 3.2 (Zakharyashev closed-domain condition). A partial Esakia morphism $f : X \rightarrow P$ satisfies ZCDC($\mathcal{D}_D$) if

$$x \notin \text{dom}(f) \implies \min f[\uparrow x] \notin \mathcal{D}_D.$$

The following is the precise dual form of the canonical-formula lemma.

Theorem 3.3 (Canonical-formula criterion). *Let A be a finite subdirectly irreducible Heyting algebra, let P be its finite rooted dual, and let $D \subseteq A^2$. For every Esakia space X ,*

$$X \not\models \rho(A, D)$$

if and only if there are a closed upset $Y \subseteq X$ and an onto cofinal partial Esakia morphism

$$f : Y \rightarrow P$$

satisfying ZCDC($\mathcal{D}_D$).

This is [4, Theorem 5.10], based on the generalized Esakia duality and the closed-domain lemma [4, Lemma 5.9]. Notice the direction: the dual object P is an onto partial image of a closed upset of X . It is not embedded into X .

4 The whole-Medvedev-frame normalization

The special self-similarity of the frames M_n removes the closed upset from [theorem 3.3](#).

Lemma 4.1 (Principal-cone normalization). *Let P be a finite rooted poset with root r . Suppose that a closed upset $Y \subseteq M_N$ carries an onto cofinal partial Esakia morphism $f : Y \rightarrow P$ satisfying a given collection of ZCDC conditions. Then, for some $m \geq 1$, there is an onto cofinal partial Esakia morphism*

$$g : M_m \rightarrow P$$

from the whole ambient Medvedev frame, satisfying the same ZCDC conditions.

Proof. Choose $x \in \text{dom}(f)$ with $f(x) = r$, using onto-ness. Since Y is an upset of M_N , it contains $\uparrow x$. Because the order is reverse inclusion,

$$\uparrow x = \{z : \emptyset \neq z \subseteq x\} \cong M_{|x|}.$$

For every $z \in \uparrow x$, its upper cone computed inside $\uparrow x$ is the same as its upper cone in M_N . Consequently all five clauses of [definition 3.1](#), cofinality, and ZCDC are inherited by the restriction $g = f \upharpoonright \uparrow x$.

Finally, (PE2) at x gives

$$g[\uparrow x] = f[\uparrow x] = \uparrow f(x) = \uparrow r = P,$$

so the restriction is still onto. $\square$

Define $E(P, D)$ to mean that, for some $n \geq 1$, there is an onto cofinal partial Esakia morphism $M_n \rightarrow P$ satisfying all the families $\mathcal{D}_{U,V}$ associated with D .

Corollary 4.2 (Exact Medvedev criterion). *For every finite rooted P and every $D \subseteq \text{Up}(P)^2$,*

$$\rho(\text{Up}(P), D) \notin \text{ML} \iff E(P, D).$$

Proof. If the formula is not in ML, it is refuted on some M_N . Apply [theorem 3.3](#) and then [lemma 4.1](#). Conversely, an $E(P, D)$ -witness is an instance of [theorem 3.3](#) with $Y = M_n$. $\square$

Here “whole frame” refers to the ambient source M_n . The partial map need not have all of M_n as its domain.

4.1 Structural completeness and direct algebraic embeddings

Structural completeness gives a second proof of the normalization and a useful algebraic formulation.

Proposition 4.3 (Direct embedding form). *Let A be finite and subdirectly irreducible. Then*

$$\rho(A, D) \notin \text{ML}$$

if and only if, for some $n \geq 1$, there is an injective map $h : A \rightarrow B_n$ preserving $0, 1, \wedge, \rightarrow$ and satisfying

$$h(a \vee b) = h(a) \vee h(b) \quad ((a, b) \in D).$$

Proof. Write $\rho(A, D) = \Delta(A, D) \rightarrow p_s$. Suppose this formula is not in ML. By structural completeness, the rule

$$\frac{\Delta(A, D)}{p_s}$$

is not admissible. Thus some substitution σ satisfies

$$\sigma\Delta(A, D) \in \text{ML}, \quad \sigma p_s \notin \text{ML}.$$

Choose n and a valuation v in B_n such that $v(\sigma p_s) < 1$, and define

$$h(a) = v(\sigma p_a).$$

Because $\sigma\Delta(A, D)$ is valid in every Medvedev algebra, its diagram equations show that h preserves the stated operations and the selected joins.

It remains to prove injectivity. If $a \neq b$, assume without loss of generality that $a \not\leq b$. Then $c = a \rightarrow b < 1$, and since s is the opremum, $c \leq s$. If $h(a) = h(b)$, preservation of implication gives

$$h(c) = h(a) \rightarrow h(b) = 1.$$

On the other hand, monotonicity and $h(s) < 1$ give

$$1 = h(c) \leq h(s) < 1,$$

a contradiction. Hence h is injective.

Conversely, such an embedding, used as the valuation $p_a \mapsto h(a)$, makes the antecedent of $\rho(A, D)$ equal to 1 and its consequent $h(s) < 1$. It therefore refutes the formula in B_n . $\square$

The generalized Esakia dual of this direct embedding is precisely the whole-source partial surjection of [corollary 4.2](#). Structural completeness of each individual logic $\text{Log}(M_n)$, proved in [7, Proposition 4], yields the sharper same-index version.

The disjunction property is not used in this reduction. In particular, it does not permit the selected domain D to be erased. The selected joins will carry the local compatibility constraints of the tiling.

5 Face semantics on a finite simplex

We record an exact combinatorial form of partial Esakia morphisms from M_n . It explains both the relation with cover-preserving maps and why a fixed- n search is finite.

Let $f : M_n \rightarrow P$ be a partial Esakia morphism. For every nonempty $S \subseteq [n]$, put

$$A_f(S) = f[\{T \in \text{dom}(f) : \emptyset \neq T \subseteq S\}], \quad \lambda_f(S) = \min A_f(S).$$

Proposition 5.1 (Exact face recurrence). *The following hold.*

- (i) $A_f(S)$ is an upset of P .
- (ii) $S \in \text{dom}(f)$ exactly when $\lambda_f(S) = \{p\}$ for some p , in which case $f(S) = p$.
- (iii) If $|S| > 1$, define

$$\mu_f(S) = \min \bigcup_{i \in S} \uparrow \lambda_f(S \setminus \{i\}).$$

Then either

$$\lambda_f(S) = \mu_f(S),$$

or

$$\lambda_f(S) = \{p\} \quad \text{and} \quad \text{Cov}(p) = \mu_f(S).$$

In the latter case S is a newly filled p -face.

- (iv) If $S \notin \text{dom}(f)$, then $\lambda_f(S)$ is nonsingleton and ZCDC is exactly the requirement $\lambda_f(S) \notin \mathcal{D}_D$.

Proof. For (i), if $f(T) = p \in A_f(S)$ and $p \leq q$, clause (PE2) supplies a domain subface $R \subseteq T \subseteq S$ with $f(R) = q$.

If $S \in \text{dom}(f)$, then (PE3) gives $A_f(S) = \uparrow f(S)$, whose unique minimum is $f(S)$. Conversely, a finite upset with the unique minimum p is exactly $\uparrow p$, so (PE3) proves (ii).

Every proper subface of S is contained in a facet $S \setminus \{i\}$. Therefore the images of all proper domain subfaces form

$$B(S) = \bigcup_{i \in S} A_f(S \setminus \{i\}) = \bigcup_{i \in S} \uparrow \lambda_f(S \setminus \{i\}).$$

If S is not a newly filled face, then $A_f(S) = B(S)$, which gives the first alternative. If $f(S) = p \notin B(S)$, the back condition says that every strict successor of p occurs on a proper subface, while all images of proper subfaces lie above p . Hence $B(S) = \uparrow p \setminus \{p\}$, whose minimal points are exactly $\text{Cov}(p)$. This proves (iii). Part (iv) follows from (ii) and [definition 3.2](#). $\square$

For fixed n, P, D , the recurrence yields a finite constraint-satisfaction problem. Decidability of $E(P, D)$ would require an effective global bound on n . No such bound exists, as a consequence of the reduction below.

6 The periodic domino problem

A Wang tile t carries four colors

$$w(t), \quad e(t), \quad s(t), \quad n(t).$$

For a finite nonempty tile set W , a finite torus tiling is a map

$$\tau : \mathbb{Z}_m \times \mathbb{Z}_\ell \longrightarrow W$$

such that, with cyclic indices,

$$\begin{aligned} e(\tau(a, b)) &= w(\tau(a + 1, b)), \\ n(\tau(a, b)) &= s(\tau(a, b + 1)). \end{aligned}$$

The problem PER asks whether such positive m, ℓ and such a τ exist.

This is the doubly periodic domino problem proved undecidable by Gurevich and Koryakov [11]; Jeandel gives a modern proof in [12]. It is recursively enumerable: enumerate m, ℓ and the finitely many maps into W .

Some sources use “periodic” for the existence of only one nonzero period. For finite Wang shifts this gives the same existence problem. Quotienting by one period turns transverse rows into a bi-infinite walk in a finite graph; a directed cycle supplies a second period. Conversely, any rank-two period lattice contains nonzero axis-parallel periods, so the tiling factors through some $\mathbb{Z}_m \times \mathbb{Z}_\ell$.

Lemma 6.1 (Surjective range). *A finite tile set W has a finite torus tiling if and only if some nonempty $W' \subseteq W$ has a finite torus tiling whose range is all of W' .*

Proof. Given a W -tiling, take W' to be its finite nonempty range. The reverse implication follows from $W' \subseteq W$. $\square$

7 The finite canonical pattern associated with a tile set

Fix a nonempty finite tile set W . We construct a finite rooted poset P_W and a set $D_W \subseteq \text{Up}(P_W)^2$.

7.1 The poset

There are four maximal *role points*

$$H_0, H_1, V_0, V_1.$$

For every role X , add two points forming the chain

$$g_X < d_X < X.$$

Add horizontal relation points

$$R_{00}, R_{01}, R_{10}, \quad \text{Cov}(R_{ab}) = \{H_0, H_1\},$$

vertical relation points

$$S_{00}, S_{01}, S_{10}, \quad \text{Cov}(S_{ab}) = \{V_0, V_1\},$$

and, for every $i, j \in \{0, 1\}$ and $t \in W$, a tile point

$$T_{ij,t}, \quad \text{Cov}(T_{ij,t}) = \{H_i, V_j\}.$$

Finally, add a root r , whose covers are every g_X , every R_{ab} , every S_{ab} , and every $T_{ij,t}$. There are no other comparabilities. In particular, the relation and tile points are covers of the root, while

$$r < g_X < d_X < X.$$

The resulting P_W is finite and rooted. Therefore $A_W = \text{Up}(P_W)$ is a finite subdirectly irreducible Heyting algebra.

Write $P_W^\circ = P_W \setminus \{r\}$ and

$$\text{Max} = \{H_0, H_1, V_0, V_1\}.$$

7.2 Pair-filling cuts

For the six pairs

$$(H_0, H_1), \quad (V_0, V_1), \quad (H_i, V_j) \quad (i, j \in \{0, 1\}),$$

place the pair of upsets

$$U = \{X\}, \quad V = \{Y\}$$

in D_W . Since X, Y are distinct maximal points, $\mathcal{D}_{U,V}$ consists of the single antichain $\{X, Y\}$.

7.3 Serial cuts

Use the following data.

direction	target Y	nonoutputs N	output Q
$H_0 \rightarrow H_1$	H_1	$\{R_{00}, R_{10}\}$	R_{01}
$H_1 \rightarrow H_0$	H_0	$\{R_{00}, R_{01}\}$	R_{10}
$V_0 \rightarrow V_1$	V_1	$\{S_{00}, S_{10}\}$	S_{01}
$V_1 \rightarrow V_0$	V_0	$\{S_{00}, S_{01}\}$	S_{10}

For each row define

$$I = P_W^\circ \setminus (\{g_Y, Q\} \cup N), \quad U = I \cup \{g_Y\}, \quad V = I \cup N,$$

and place (U, V) in D_W . Thus

$$U \setminus V = \{g_Y\}, \quad V \setminus U = N, \quad P_W \setminus (U \cup V) = \{r, Q\}.$$

In particular $d_Y \in U \cap V$.

7.4 Compatibility cuts

Put $R_0 = R_{01}$ and $R_1 = R_{10}$. For $i, j \in \{0, 1\}$ and $t \in W$, let

$$G_{ij,t}^H = \{T_{1-i,j,t'} : e(t) = w(t')\}.$$

Define

$$\begin{aligned} I_{ij,t}^H &= P_W^\circ \setminus (\{R_i, T_{ij,t}, d_{H_i}, g_{H_i}\} \cup G_{ij,t}^H), \\ U_{ij,t}^H &= I_{ij,t}^H \cup \{R_i\}, \\ V_{ij,t}^H &= I_{ij,t}^H \cup \{T_{ij,t}\}. \end{aligned}$$

Place $(U_{ij,t}^H, V_{ij,t}^H)$ in D_W .

Likewise put $S_0 = S_{01}$, $S_1 = S_{10}$, and

$$G_{ij,t}^V = \{T_{i,1-j,t'} : n(t) = s(t')\}.$$

Set

$$\begin{aligned} I_{ij,t}^V &= P_W^\circ \setminus (\{S_j, T_{ij,t}, d_{V_j}, g_{V_j}\} \cup G_{ij,t}^V), \\ U_{ij,t}^V &= I_{ij,t}^V \cup \{S_j\}, \\ V_{ij,t}^V &= I_{ij,t}^V \cup \{T_{ij,t}\}, \end{aligned}$$

and add these pairs to D_W .

For a horizontal cut, the exclusive points are R_i and $T_{ij,t}$. The points outside the union are

$$\{r, d_{H_i}, g_{H_i}\} \cup G_{ij,t}^H.$$

The vertical statement is analogous.

Lemma 7.1 (Legitimacy of the domain). *Every set U, V used above is an upset of P_W . Hence D_W is a genuine finite subset of A_W^2 .*

Proof. Singletons of maximal points are upsets. For a serial cut, the elements removed from P_W° are g_Y and relation points covering the root. No point retained in I lies below one of them. Moreover, $d_Y, Y \in I$, so adjoining g_Y preserves upward closure, while the maximal successors of every point of N already belong to I .

For a compatibility cut, the binary relation and tile points that are removed all cover the root. The only additional removed chain points are g_X, d_X together. Thus no retained point lies below a removed one. Adjoining R_i or $T_{ij,t}$ preserves upward closure because their maximal successors remain in I . $\square$

8 Soundness: a partial morphism yields a torus

Assume $E(P_W, D_W)$, witnessed by an onto cofinal partial Esakia morphism

$$f : M_n \rightarrow P_W.$$

8.1 Roles of domain faces

Every singleton of M_n belongs to the domain. Indeed, cofinality applied to $\{a\}$ can only choose the nonempty subface $\{a\}$. Its image is a maximal point of P_W , because its upper cone in M_n contains only itself. We call a an X -vertex if $f(\{a\}) = X \in \text{Max}$.

Lemma 8.1 (Support-role lemma). *If $F \in \text{dom}(f)$ and $f(F) = p$, then the set of role labels of the vertices of F is exactly*

$$\text{Max}(\uparrow p).$$

In particular:

- (a) a face labelled g_X or d_X consists solely of X -vertices;
- (b) a face labelled R_{ab} contains only H_0, H_1 -vertices and contains both roles;
- (c) a face labelled S_{ab} has the analogous vertical property;
- (d) a face labelled $T_{ij,t}$ contains only H_i, V_j -vertices and contains both roles;
- (e) a face labelled r contains all four roles.

Proof. For every $a \in F$, the singleton $\{a\}$ is a domain subface, so its maximal label belongs to

$$f[\uparrow F] = \uparrow p.$$

Conversely, if X is a maximal point above p , (PE2) gives a domain subface $G \subseteq F$ labelled X . Since $f[\uparrow G] = \uparrow X = \{X\}$, every vertex of G has role X . Thus F contains an X -vertex. $\square$

8.2 Every mixed pair has a unique color

Lemma 8.2 (Pair filling). *Let x, y have one of the six designated role pairs*

$$(H_0, H_1), \quad (V_0, V_1), \quad (H_i, V_j).$$

Then $\{x, y\}$ is a domain face. It is labelled by exactly one of the corresponding R -, S -, or T -points.

Proof. If $\{x, y\}$ were outside the domain, the images of its proper nonempty subfaces would have frontier $\{X, Y\}$, forbidden by the pair-filling cut. Thus it is a domain face. If its label is p , then

$$f[\uparrow \{x, y\}] = \uparrow p.$$

The support-role lemma and the definition of P_W show that the only possible p is one of the designated binary points over X, Y . Uniqueness follows because f is a function. $\square$

We henceforth speak of the color of such a mixed edge.

8.3 The guards force directed cycles

Onto-ness supplies, for each role X , a domain face G_X with $f(G_X) = g_X$. By lemma 8.1, it consists only of X -vertices. Applying (PE2) first to d_X and then to X gives strictly nested domain faces

$$G_X \supsetneq D_X \supsetneq Z_X, \quad f(D_X) = d_X, \quad f(Z_X) = X.$$

Consequently $|G_X| \geq 3$.

Lemma 8.3 (Seriality). *For every $x \in G_{H_i}$, some $y \in G_{H_{1-i}}$ makes the edge $\{x, y\}$ have color R_i . Likewise, for every $x \in G_{V_j}$, some $y \in G_{V_{1-j}}$ makes $\{x, y\}$ have color S_j .*

Proof. We prove one of the four identical cases. Let $X \rightarrow Y$ be the selected direction, with output Q and nonoutputs N , and fix $x \in G_X$. Put

$$F = G_Y \cup \{x\}.$$

Suppose that every edge $\{x, y\}$, $y \in G_Y$, has a color in N .

First, Q cannot occur as the image of any domain subface of F . For if $K \subseteq F$ were labelled Q , the support-role lemma would give vertices of both roles in K . Their two-face is domain by lemma 8.2; its binary label belongs to $\uparrow Q$, and therefore equals Q , contrary to the assumption. The root r cannot occur either, because F has only the two roles X, Y , whereas an r -face contains all four roles.

The face F is not a domain face. Indeed, its image upset contains g_Y and at least one point of N . The only common lower bound of these two root covers is r , but an r -label has just been excluded. Consequently its frontier lies in

$$P_W \setminus \{r, Q\} = U \cup V.$$

Moreover g_Y is a minimal active point, since its only strict lower point is r , and every active nonoutput is minimal for the same reason. Thus the frontier meets both $U \setminus V = \{g_Y\}$ and $V \setminus U = N$. It belongs to $\mathcal{D}_{U,V}$, contradicting ZCDC. Hence an output edge exists. $\square$

Define a directed bipartite graph on $G_{H_0} \cup G_{H_1}$ by

$$x \longrightarrow y \iff \begin{cases} x \in G_{H_0}, y \in G_{H_1}, & \{x, y\} \text{ has color } R_{01}, \\ x \in G_{H_1}, y \in G_{H_0}, & \{x, y\} \text{ has color } R_{10}. \end{cases}$$

Every vertex has an outgoing edge, so finiteness yields a directed alternating cycle. The analogous construction yields a directed vertical cycle.

8.4 Compatibility on cycle products

Let $x \rightarrow y$ be an edge of the horizontal cycle, where x has role H_i , and let v be a vertex of the vertical cycle having role V_j . By pair filling, for unique tiles $t, t' \in W$,

$$f(\{x, v\}) = T_{ij,t}, \quad f(\{y, v\}) = T_{1-i,j,t'}.$$

Lemma 8.4 (Horizontal compatibility). *For the preceding data,*

$$e(t) = w(t').$$

Proof. The three two-faces of $F = \{x, y, v\}$ have the labels

$$R_i, \quad T_{ij,t}, \quad T_{1-i,j,t'}.$$

The face F is not in the domain. Its image on proper subfaces has three distinct minimal root covers, and their only common lower bound is r . But an r -face must contain all four roles, while F contains only H_0, H_1, V_j . Since every proper nonempty subface is either a singleton or one of the three mixed pairs, the frontier is exactly

$$\{R_i, T_{ij,t}, T_{1-i,j,t'}\}.$$

If t' were horizontally incompatible with t , its target point would belong to $I_{ij,t}^H$. The displayed frontier would then be contained in $U_{ij,t}^H \cup V_{ij,t}^H$ and would meet the two exclusive singleton parts R_i and $T_{ij,t}$. This contradicts ZCDC. $\square$

The vertical compatibility lemma is identical, giving $n(t) = s(t')$ along every vertical cycle edge.

Index the horizontal and vertical directed cycles cyclically. Assign to a pair of cycle vertices (x, v) the unique tile t for which the mixed edge $\{x, v\}$ has label $T_{ij,t}$, with i, j determined by the two roles. The horizontal and vertical compatibility lemmas show that this is a finite torus tiling by W . We have proved:

Proposition 8.5 (Soundness). *If $E(P_W, D_W)$, then W has a finite torus tiling.*

9 Completeness: a torus yields a partial morphism

Suppose now that

$$\tau : \mathbb{Z}_m \times \mathbb{Z}_\ell \longrightarrow W$$

is a finite torus tiling whose range is all of W . Put

$$M = 3m, \quad L = 3\ell,$$

and extend τ periodically to $\mathbb{Z}_M \times \mathbb{Z}_L$.

Take four pairwise disjoint coordinate blocks

$$H_i = \{h_i(a) : a \in \mathbb{Z}_M\}, \quad V_j = \{v_j(b) : b \in \mathbb{Z}_L\},$$

and let

$$\Omega = H_0 \sqcup H_1 \sqcup V_0 \sqcup V_1.$$

We shall define the partial morphism on $M_\Omega = (\mathcal{P}^*(\Omega), \supseteq)$.

9.1 Calibrated supports

For each $p \in P_W$, define a nonempty family $\Sigma_p \subseteq \mathcal{P}^*(\Omega)$ as follows.

- (S1) For a role X , every singleton from the block X belongs to Σ_X .
- (S2) Every two-element subset of a role block X belongs to Σ_{d_X} .
- (S3) The whole block X is the unique member of Σ_{g_X} .
- (S4) For $i \in \{0, 1\}$ and $a \in \mathbb{Z}_M$, put

$$\{h_i(a), h_{1-i}(a+1)\} \in \Sigma_{R_i}.$$

Every remaining H_0 - H_1 pair belongs to $\Sigma_{R_{00}}$.

- (S5) For $j \in \{0, 1\}$ and $b \in \mathbb{Z}_L$, put

$$\{v_j(b), v_{1-j}(b+1)\} \in \Sigma_{S_j}.$$

Every remaining V_0 - V_1 pair belongs to $\Sigma_{S_{00}}$.

- (S6) For every i, j, a, b , put

$$\{h_i(a), v_j(b)\} \in \Sigma_{T_{ij, \tau(a \bmod m, b \bmod \ell)}}.$$

- (S7) Put $\Omega \in \Sigma_r$.

Because $M, L \geq 3$, the two directed successor matchings in each pair of phase blocks are disjoint and default R_{00} - and S_{00} -pairs exist. Since τ is onto W , every tile point has a support in each of its four phases. Hence every $p \in P_W$ has a support.

For a nonempty $S \subseteq \Omega$, define

$$\text{Act}(S) = \{p \in P_W : \text{some } H \in \Sigma_p \text{ satisfies } H \subseteq S\}.$$

Lemma 9.1 (Calibration). *For every $p \in P_W$ and every $H \in \Sigma_p$,*

$$\text{Act}(H) = \uparrow p.$$

Proof. The verification follows directly from the types of supports.

- A singleton role support activates exactly its maximal role.
- A same-role pair activates exactly d_X and X . It does not contain the whole role block, since block size is at least three.

- A whole role block activates exactly g_X, d_X, X .
- A horizontal relation support activates its unique relation label and the two horizontal roles; a vertical relation support is analogous.
- A cell support activates its unique tile point and its horizontal and vertical roles.
- The full set Ω activates every point of P_W , and hence exactly $\uparrow r = P_W$.

No support of another type fits into any of the proper supports listed above. These sets are precisely the principal upsets determined by the definition of P_W . $\square$

We use a general support lemma.

Lemma 9.2 (Support realization). *Let P and V be finite. Suppose nonempty support families $\Sigma_p \subseteq \mathcal{P}^*(V)$ satisfy*

$$\{q : \exists K \in \Sigma_q (K \subseteq H)\} = \uparrow p \quad (H \in \Sigma_p).$$

For nonempty $S \subseteq V$, define $\text{Act}(S)$ by support containment, and define a partial map by

$$S \in \text{dom}(f), f(S) = p \iff \text{Act}(S) = \uparrow p.$$

Then $f : M_V \rightarrow P$ is an onto partial Esakia morphism. If every singleton is a calibrated support, it is cofinal.

Proof. First, $\text{Act}(S)$ is an upset. If a support $H \subseteq S$ activates $p \leq q$, calibration of H supplies a q -support contained in H , and hence in S .

The central identity is

$$f[\uparrow S] = \text{Act}(S). \tag{1}$$

For the left-to-right inclusion, if $T \subseteq S$ is a domain face with $f(T) = p$, then $p \in \text{Act}(T) \subseteq \text{Act}(S)$. Conversely, if $p \in \text{Act}(S)$, choose $H \in \Sigma_p$ with $H \subseteq S$. Calibration gives $\text{Act}(H) = \uparrow p$, so $H \in \text{dom}(f)$ and $f(H) = p$. Thus $p \in f[\uparrow S]$.

Now suppose $S, T \in \text{dom}(f)$ and $S \leq T$, which means $T \subseteq S$. Then

$$\uparrow f(T) = \text{Act}(T) \subseteq \text{Act}(S) = \uparrow f(S),$$

so $f(S) \leq f(T)$, proving (PE1). If $f(S) = p \leq q$, the identity (1) supplies a domain subface $H \subseteq S$ with $f(H) = q$, proving (PE2). Clause (PE3) is the domain definition together with (1). Clause (PE4) holds by finiteness.

For an upset $U \subseteq P$,

$$M_V \setminus \downarrow f^{-1}(P \setminus U) = \{S : \text{Act}(S) \subseteq U\}.$$

This is an upset of M_V , since $S \leq T$ implies $\text{Act}(T) \subseteq \text{Act}(S)$, and it is clopen by finiteness. This proves (PE5).

Every p is attained on each member of Σ_p , proving onto-ness. If every singleton is a domain face, then every nonempty S has a singleton subface T , so $S \leq T \in \text{dom}(f)$; this is cofinality. $\square$

Apply lemma 9.2 using lemma 9.1. We obtain an onto cofinal partial Esakia morphism

$$f : M_\Omega \rightarrow P_W.$$

9.2 The global ZCDC verification

We must check all non-domain faces, not only the displayed supports.

Lemma 9.3 (Outside-point principle). *Let $U, V \subseteq P_W$ be upsets and $S \subseteq \Omega$. If some $q \in \text{Act}(S)$ lies outside $U \cup V$, then $\min \text{Act}(S)$ has a point outside $U \cup V$.*

Proof. Choose a minimal active $p \leq q$. The complement of the upset $U \cup V$ is a downset. Since q is in this complement and $p \leq q$, p is also in the complement. $\square$

Let $S \not\subseteq \text{dom}(f)$ and put $C = \min \text{Act}(S)$.

For a pair-filling cut, suppose $C \in \mathcal{D}_{\{X\},\{Y\}}$. Then $C = \{X, Y\}$, so S contains vertices of both roles. Their mixed pair has a relation or tile support q below X, Y , and $q \notin \{X, Y\}$. The outside-point principle contradicts $C \subseteq \{X, Y\}$.

For a serial cut, suppose $C \in \mathcal{D}_{U,V}$. Since the exclusive parts are $\{g_Y\}$ and N , the points g_Y and some $q \in N$ are active. Activation of g_Y means that the entire target block Y is contained in S . A support for $q \in N$ contains a source-role vertex x . The prescribed successor of x in the complete target block gives a Q -support contained in S . Since $Q \notin U \cup V$, the outside-point principle contradicts the assumed forbidden frontier.

For a horizontal compatibility cut, a forbidden frontier activates both R_i and $T_{ij,t}$. Choose supports

$$\{a, b\} \in \Sigma_{R_i}, \quad \{c, v\} \in \Sigma_{T_{ij,t}},$$

where $a, c \in H_i$, $b \in H_{1-i}$, and $v \in V_j$.

If $a \neq c$, the same-role pair $\{a, c\}$ activates d_{H_i} , which lies outside the compatibility union. If $a = c$, then b is the prescribed horizontal successor of a . The pair $\{b, v\}$ activates

$$T_{1-i,j,t'} \in G_{ij,t}^H,$$

because the original τ satisfies horizontal compatibility. This point also lies outside the compatibility union. Both cases contradict the outside-point principle. The vertical compatibility cuts are identical, using d_{V_j} and the north-south condition.

We have checked every selected pair and every non-domain S , so f satisfies ZCDC. Therefore:

Proposition 9.4 (Completeness). *If W has a finite torus tiling using every tile of W , then $E(P_W, D_W)$.*

10 The undecidability reduction

For each nonempty $W' \subseteq W$, form the finite rooted poset $P_{W'}$, the finite domain $D_{W'}$, and the canonical formula

$$\rho_{W'} = \rho(\text{Up}(P_{W'}), D_{W'}).$$

Theorem 10.1 (Tiling equivalence). *For every finite nonempty Wang tile set W ,*

$$W \in \text{PER} \iff \exists \emptyset \neq W' \subseteq W \quad \rho_{W'} \notin \text{ML}.$$

Proof. Suppose first that $W \in \text{PER}$. By lemma 6.1, some nonempty $W' \subseteq W$ has a surjective finite torus tiling. Proposition 9.4 gives $E(P_{W'}, D_{W'})$, and corollary 4.2 gives $\rho_{W'} \notin \text{ML}$.

Conversely, if $\rho_{W'} \notin \text{ML}$, then corollary 4.2 yields $E(P_{W'}, D_{W'})$. Proposition 8.5 gives a finite torus tiling by W' , which is also a tiling by W . $\square$

Rename variables between different conjuncts if necessary, and define the single formula

$$\Phi_W = \bigwedge_{\emptyset \neq W' \subseteq W} \rho_{W'}.$$

This construction is effective. Its exponential size is irrelevant to computability.

Corollary 10.2 (Many-one reduction). *For every finite nonempty W ,*

$$W \in \text{PER} \iff \Phi_W \notin \text{ML}.$$

Consequently non-theoremhood in ML, and hence theoremhood in ML, is undecidable.

Proof. A conjunction belongs to a logic exactly when every conjunct belongs to the logic. Apply theorem 10.1. Since $W \mapsto \Phi_W$ is a total computable map and PER is undecidable, the conclusion follows. $\square$

For the problem restricted to one canonical formula, the same construction is a finite nonadaptive truth-table reduction: query the finitely many patterns $\rho_{W'}$ and accept exactly when at least one is a non-theorem. We do not claim that Φ_W itself is a single canonical formula.

11 Computability-theoretic consequences

Proof of theorem 2.1. Nonmembership in ML is recursively enumerable. Given a formula φ , enumerate $n = 1, 2, \dots$ and, for each n , the finitely many valuations of the variables of φ in the finite algebra B_n . This process halts exactly when it finds a refutation. Thus ML is co-recursively enumerable.

By corollary 10.2, ML is undecidable. If ML were also recursively enumerable, dovetailing enumerations of membership and nonmembership would decide it. Hence ML is not recursively enumerable.

The deductive closure of any recursively enumerable set of axioms under the usual effective finitary intuitionistic calculus is recursively enumerable. It follows that ML has no recursively enumerable axiomatization. $\square$

Let

$$\text{Can}(\text{ML}) = \{\langle P, D \rangle : P \text{ is finite rooted, } D \subseteq \text{Up}(P)^2, \rho(\text{Up}(P), D) \in \text{ML}\}.$$

Proof of theorem 2.2. The complement of $\text{Can}(\text{ML})$ is recursively enumerable by the same finite-countermodel search. Undecidability follows from the finite truth-table reduction in theorem 10.1.

Finally, the canonical-formula axiomatization theorem says that every intermediate logic, in particular ML, is axiomatized by a subset of the canonical formulas it contains [4, Theorem 4.9]. Consequently all of $\text{Can}(\text{ML})$ also axiomatize ML. If $\text{Can}(\text{ML})$ were recursively enumerable, this would give a recursively enumerable axiomatization of ML, contradicting [theorem 2.1](#). Hence it is co-recursively enumerable but not recursively enumerable. $\square$

Corollary 11.1 (No computable dimension bound). *There is no total computable function $b(P, D)$ with the property that whenever $E(P, D)$ holds, it has a witness on some M_n with $n \leq b(P, D)$.*

Proof. Such a function would decide $E(P, D)$: compute the bound and exhaustively test the finitely many partial maps on $M_1, \dots, M_{b(P, D)}$. Applying this procedure to all nonempty $W' \subseteq W$ would decide PER, by [theorem 10.1](#). $\square$

Thus no general folding, finite-profile, or terminating support-chase theorem can yield an effective cutoff for the canonical-pattern problem. Fixed dimensions remain finite CSPs, but the least required dimension encodes unbounded periodic-tiling information.

12 The Jankov-generated core

We use the modern spelling *Jankov*; “Jankov logic” is avoided because that name commonly denotes KC. If P is a finite rooted poset, write $J(P)$ for its Jankov–de Jongh formula. When $A = \text{Up}(P)$,

$$\text{IPC} \vdash J(P) \leftrightarrow \rho(A, A^2).$$

Thus a Jankov formula is the full-join-domain special case of a bounded canonical formula. Define the *Jankov-generated core* of an intermediate logic L by

$$\text{Jan}(L) = \text{IPC} + \{J(P) : P \text{ is finite rooted and } J(P) \in L\},$$

and retain the notation requested in the motivating question:

$$Y_{\text{ML}} = \text{Jan}(\text{ML}).$$

Proposition 12.1 (Finite frames of the Jankov core). *For every intermediate logic L and every finite rooted P ,*

$$J(P) \in L \iff P \not\models L.$$

Consequently,

$$\text{FinFrm}(\text{Jan}(L)) = \text{FinFrm}(L).$$

Moreover, $\text{Jan}(L)$ is the greatest sublogic of L axiomatizable by Jankov formulas, equivalently the greatest join-splitting sublogic of L .

Proof. The frame P refutes $J(P)$, so the forward implication is immediate. Conversely, suppose $P \not\models L$, and put $A = \text{Up}(P)$. If $J(P) \notin L$, completeness supplies an L -algebra B refuting $J(P)$. The Jankov lemma gives $A \in \mathbf{ISH}(B)$, so A is also an L -algebra, contrary to $P \not\models L$. This proves the equivalence. If a finite P is not an L -frame, its own formula $J(P)$ is an axiom of $\text{Jan}(L)$ and is refuted on P ; this proves the finite-frame equality. The final assertion is the splitting theorem for Jankov formulas [4, Theorem 3.9]. $\square$

Since $\mathbf{ML}$ has the finite-model property, the fmp span attached to its finite frames has lower and upper endpoints

$$\mathbf{ML}^- = Y_{\mathbf{ML}}, \quad \mathbf{ML}^+ = \text{Log}(\text{FinFrm}(\mathbf{ML})) = \mathbf{ML}.$$

Thus $Y_{\mathbf{ML}} = \mathbf{ML}$ is exactly the assertion that this span collapses, or equivalently that $\mathbf{ML}$ is Jankov-axiomatizable [5].

Proposition 12.2 (Exact finite-frame recognition problem). *For every finite rooted poset P ,*

$$P \models \mathbf{ML} \iff \exists n \geq 1 M_n \twoheadrightarrow_p P.$$

Therefore, if

$$I_{\mathbf{ML}} = \{P : P \text{ is finite rooted and } J(P) \in \mathbf{ML}\},$$

then

$$P \notin I_{\mathbf{ML}} \iff \exists n \geq 1 M_n \twoheadrightarrow_p P,$$

and $I_{\mathbf{ML}}$ is co-recursively enumerable.

Proof. The Jankov–de Jongh theorem says that a frame refutes $J(P)$ exactly when a generated subframe has P as an onto p -morphic image. Every generated cone of M_n is another finite Medvedev frame. Hence

$$J(P) \notin \mathbf{ML} \iff \exists n M_n \twoheadrightarrow_p P.$$

Combine this with [proposition 12.1](#). The condition on the right is recursively enumerable: enumerate n and exhaustively test the finitely many maps $M_n \rightarrow P$. $\square$

Theorem 12.3 (Exact effectivity equivalence). *The following statements are equivalent.*

- (i) $Y_{\mathbf{ML}}$ is recursively enumerable.
- (ii) $I_{\mathbf{ML}}$ is recursively enumerable.
- (iii) $I_{\mathbf{ML}}$ is recursive.
- (iv) Recognition of the finite rooted $\mathbf{ML}$ -frames is decidable.

If these conditions hold, then $Y_{\mathbf{ML}} \subsetneq \mathbf{ML}$.

Proof. For every finite rooted P ,

$$J(P) \in Y_{\mathbf{ML}} \iff J(P) \in \mathbf{ML}.$$

Thus an enumeration of $Y_{\mathbf{ML}}$ semidecides $I_{\mathbf{ML}}$. Conversely, an enumeration of $I_{\mathbf{ML}}$ effectively enumerates the Jankov axioms and therefore their deductive closure. By [proposition 12.2](#), $I_{\mathbf{ML}}$ is already co-r.e.; it is r.e. exactly when it is recursive, and its complement is exactly the set of finite rooted $\mathbf{ML}$ -frames. Finally, $\mathbf{ML}$ is not r.e. by [theorem 2.1](#), whereas $Y_{\mathbf{ML}}$ would be r.e. $\square$

12.1 The unresolved finite-folding obstruction

[Theorem 12.3](#) is a reduction, not a decision procedure. Neither the canonical tiling construction nor structural completeness and the disjunction property decide finite ML -frame recognition. In particular, the present argument establishes neither

$$Y_{\text{ML}} \neq \text{ML} \quad \text{nor} \quad Y_{\text{ML}} = \text{ML}.$$

If equality held, I_{ML} would be co-r.e. but not r.e.; this is fully compatible with all computability results above.

The exact obstruction can be expressed using Skvortsov's infinite frame. For a finite rooted P , consider the finite-folding implication

$$M_I \rightarrow_p P \text{ for some nonempty } I \subseteq \mathbb{N} \implies M_n \rightarrow_p P \text{ for some finite } n. \quad (2)$$

Shehtman proved (2) when P is *duplicate-free*, meaning that no point has exactly one immediate successor. More precisely, if r is the root and $d_P(r)$ is its depth, his Lemma 17 supplies the explicit source

$$M_{2^{d_P(r)} - 1}.$$

His Proposition 18 and Corollary 19 consequently identify the finite-base and arbitrary-base Jankov counterframes on this subclass [21]. Immediately afterward he notes that arbitrary finite frames remain unhandled and that the depth bound of Lemma 17 fails in general. Thus the duplicate-free theorem cannot be quoted as a general finite-folding result. In the same discussion Shehtman calls the equality $Y_{\text{ML}} = \text{ML}$ χ -completeness and records both that question and general finite-frame recognition as unresolved [21, §6].

There is also an elementary reason why restriction to finitely many coordinates is inadequate. On the two-point chain $r < t$, the map

$$f(A) = \begin{cases} r, & A \subseteq \mathbb{N} \text{ is infinite,} \\ t, & A \subseteq \mathbb{N} \text{ is finite and nonempty} \end{cases}$$

is an onto p -morphism $M_{\mathbb{N}} \rightarrow \{r < t\}$, but no finite restriction contains an r -labelled face. The chain has other finite realizations, so this is not a counterexample to (2); it shows that a proof would require a genuine finite re-realization rather than compact restriction.

There is nevertheless a useful finite quotient common to every total witness. It isolates where a finite-folding proof would have to do new work.

Proposition 12.4 (The maximal-support quotient). *Let P be finite and rooted and let $f : M_I \rightarrow_p P$, where $I \neq \emptyset$ may be infinite. Put $K = \text{Max}(P)$, $c(i) = f(\{i\})$, and*

$$T(p) = \text{Max}(\uparrow p) \quad (p \in P).$$

Then $c(i) \in K$,

$$T(f(A)) = \{c(i) : i \in A\} \quad (\emptyset \neq A \subseteq I), \quad (\heartsuit)$$

and $T : P \rightarrow_p M_K$ is an onto p -morphism.

Proof. The singleton $\{i\}$ is maximal in M_I . The back condition at $\{i\}$ therefore makes $c(i)$ maximal in P . If $i \in A$, then $A \leq \{i\}$, so $c(i)$ is a maximal point above $f(A)$. Conversely,

if $m \in \text{Max}(\uparrow f(A))$, the back condition supplies a nonempty $B \subseteq A$ with $f(B) = m$. For $i \in B$, $m = f(B) \leq f(\{i\}) = c(i)$; maximality gives $m = c(i)$. This proves $(\heartsuit)$.

Finiteness of P makes every $T(p)$ nonempty. If $p \leq q$, then $T(q) \subseteq T(p)$, which is $T(p) \leq T(q)$ in the reverse-inclusion order of M_K . For the back condition, let $\emptyset \neq U \subseteq T(p)$, choose A with $f(A) = p$, and, for each $m \in U$, choose $i_m \in A$ with $c(i_m) = m$. With $B = \{i_m : m \in U\}$, one has $B \subseteq A$, hence $f(B) \geq p$, and $(\heartsuit)$ gives $T(f(B)) = U$. Finally, onto-ness of f shows that every maximal color occurs; choosing one representative of each color in a nonempty $U \subseteq K$ and applying $(\heartsuit)$ proves that T is onto. $\square$

Thus finite and infinite witnesses have the same Boolean support quotient. The unresolved information lies in the fibers

$$P_U = \{p \in P : T(p) = U\}.$$

Selecting finitely many color representatives reconstructs M_K , but it does not reconstruct these fibers. The obstruction has an equivalent colored-Boolean-algebra form. For fixed P , color every nonzero element of $\mathcal{P}(I)$ by predicates C_p , with $C_p(A)$ meaning $f(A) = p$. Being a total onto p -morphism is expressed by finitely many conditions: the C_p 's partition the nonzero elements and every color occurs;

$$C_p(A) \wedge 0 < B \subseteq A \wedge C_q(B) \implies p \leq q;$$

and, for each $q \geq p$,

$$C_p(A) \implies \exists B (0 < B \subseteq A \wedge C_q(B)).$$

Thus full finite folding asks whether existence of such a coloring on any full powerset algebra already implies existence on a finite Boolean algebra, which is itself a finite powerset algebra. A naive witness-closure construction need not terminate: after adding a strict-successor witness, a complementary piece may retain the old label and repeat the same obligation. This is the residual unary-duplication obstruction that the duplicate-free theorem avoids.

Ordinary structural completeness is useful in [proposition 4.3](#), but it concerns single-conclusion admissibility. The disjunction property is naturally a multiple-conclusion admissible rule, and the two hypotheses do not complete a selected partial diagram into a full Heyting diagram [9]. In particular, no theorem

$$\text{structural completeness} + \text{disjunction property} \implies \text{Jankov axiomatizability}$$

is available here. Maximality of ML among logics with the disjunction property concerns extensions of ML , while $Y_{\text{ML}} \subseteq \text{ML}$ is a sublogic.

13 The infinite Medvedev frame and Skvortsov's logic

For every nonempty set I , put

$$M_I = (\mathcal{P}(I) \setminus \{\emptyset\}, \supseteq).$$

Thus $S \leq T$ means $T \subseteq S$, and $\uparrow S \cong M_S$. Skvortsov's logic of infinite problems is

$$\text{Skv} = \text{Log}(M_{\mathbb{N}}).$$

Skvortsov proved that this is also the intersection of the logics of $\mathcal{A}^0 = (\mathcal{A} \setminus \{\emptyset\}, \supseteq)$, where $\mathcal{A}$ ranges over infinite atomic union-semilattices of sets; the same intersection results from restricting to atomic lattices or atomic Boolean algebras [22]. He proved that $\mathbf{Skv}$ is recursively axiomatizable. This is enumerability, not decidability: Shehtman still listed both the properness of $\mathbf{Skv} \subseteq \mathbf{ML}$ and the decidability of $\mathbf{Skv}$ as open [21].

The finite-cofinite algebra

$$\mathbf{FC}(\mathbb{N}) = \{S \subseteq \mathbb{N} : S \text{ is finite or } \mathbb{N} \setminus S \text{ is finite}\}$$

is the least infinite atomic Boolean algebra of sets, whereas $\mathcal{P}(\mathbb{N})$ is the greatest. Skvortsov's theorem gives

$$\mathbf{Skv} \subseteq \mathbf{Log}(\mathbf{FC}(\mathbb{N}) \setminus \{\emptyset\}, \supseteq) \subseteq \mathbf{ML}. \quad (3)$$

It does not identify either inclusion as equality. Moreover, the construction below uses four disjoint infinite, non-cofinite supports and therefore says nothing about the middle logic in (3).

13.1 A raw-frame canonical criterion

One cannot apply the finite/discrete Esakia-space criterion directly to $M_{\mathbb{N}}$: the infinite discrete space is not compact. We use the following purely Kripke-algebraic replacement. A *raw partial Esakia morphism* below is a partial map satisfying (PE1)–(PE3) and cofinality from Definition 3.1; the topological clauses (PE4)–(PE5) are omitted.

Theorem 13.1 (Raw-frame canonical criterion). *Let X be any poset, let P be a finite rooted poset, put $A = \mathbf{Up}(P)$, and let $D \subseteq A^2$. The following are equivalent.*

- (i) $X \not\models \rho(A, D)$.
- (ii) *Some generated cone $Y = \uparrow x_0$ admits an injective map $h : A \rightarrow \mathbf{Up}(Y)$ preserving $0, 1, \wedge, \rightarrow$, and satisfying*

$$h(U \cup V) = h(U) \cup h(V) \quad ((U, V) \in D).$$

- (iii) *Some generated cone $Y = \uparrow x_0$ admits an onto cofinal raw partial Esakia morphism $f : Y \rightarrow P$ satisfying $\mathbf{ZCDC}(\mathcal{D}_D)$.*

Proof. Write $s = P \setminus \{r\}$ for the opremum of A , where r is the root, and write $\rho(A, D) = \Delta(A, D) \rightarrow p_s$.

If a valuation refutes this implication, some successor x_0 forces $\Delta(A, D)$ and does not force p_s . On $Y = \uparrow x_0$, let $h(a)$ be the truth set of p_a . The diagram equations hold throughout Y , so h has the preservation properties in (ii), and $h(s) \neq Y$. It is injective: if $a \not\leq b$, then $c = a \rightarrow b < 1$, hence $c \leq s$; but $h(a) = h(b)$ would give

$$Y = h(c) \subseteq h(s) \neq Y.$$

Conversely, such an h , used as the valuation of the diagram variables, refutes $\rho(A, D)$. This proves (i) $\iff$ (ii).

Suppose h is as in (ii). For $x \in Y$, put

$$F_x = \{a \in A : x \in h(a)\}.$$

This is a proper lattice filter of A , and $x \leq y$ implies $F_x \subseteq F_y$. Since preservation of meets makes h monotone, $a \leq b$ implies $h(a) \subseteq h(b)$. A point $p \in P$ corresponds to the prime filter

$$F_p = \{U \in \text{Up}(P) : p \in U\}.$$

Declare $x \in \text{dom}(f)$ exactly when $F_x = F_p$ for some p , and then put $f(x) = p$.

We first prove the finite-type realization fact

$$Q \supseteq F_x, Q \text{ prime} \implies \exists y \geq x (F_y = Q). \quad (4)$$

Put $q = \bigwedge Q$ and $b = \bigvee(A \setminus Q)$. If the conclusion failed, then every $y \geq x$ with $y \in h(q)$ would satisfy $Q \subseteq F_y$, because $Q = \uparrow q$. As $F_y \neq Q$, some $a \notin Q$ would lie in F_y ; monotonicity and $a \leq b$ would then give $y \in h(a) \subseteq h(b)$. The Kripke clause for implication would give

$$x \in h(q) \rightarrow h(b) = h(q \rightarrow b).$$

Thus $q \rightarrow b \in F_x \subseteq Q$. Since $q \in Q$, closure of the filter under modus ponens gives $b \in Q$. Finite primeness would then put some member of $A \setminus Q$ in Q , a contradiction. This proves (4).

Consequently, for all $x \in Y$,

$$f[\uparrow x] = \{p \in P : F_x \subseteq F_p\}. \quad (5)$$

Every prime filter of the finite algebra $\text{Up}(P)$ is F_p for a unique $p \in P$, and every proper filter has a prime extension. Thus this set is nonempty. It is an upset; if $F_x = F_p$, it is $\uparrow p$. Conversely, if it is $\uparrow p$, then the prime-filter separation theorem says that the intersection of all prime extensions of F_x is F_x ; the corresponding intersection for $\uparrow p$ is F_p . Hence $F_x = F_p$. Equation (5) proves (PE1)–(PE3) and cofinality. As $h(s) < Y$, some $x \notin h(s)$. Every $a < 1$ satisfies $a \leq s$, so monotonicity gives $F_x = \{1\} = F_r$; applying (5) there proves onto-ness.

For ZCDC, take $x \notin \text{dom}(f)$, put $C = \min f[\uparrow x]$, and suppose $C \in \mathcal{D}_{U,V}$ for $(U, V) \in D$. Then $f[\uparrow x] \subseteq U \cup V$, whence $U \cup V \in F_x$. Indeed, otherwise the finite prime-filter theorem and (4) would produce a member of $f[\uparrow x]$ outside $U \cup V$. Selected-join preservation gives

$$x \in h(U \cup V) = h(U) \cup h(V).$$

Thus $f[\uparrow x] \subseteq U$ or $f[\uparrow x] \subseteq V$, contrary to the two exclusive intersections required in $\mathcal{D}_{U,V}$. This proves (ii) $\Rightarrow$ (iii).

Conversely, suppose f witnesses (iii), abbreviate $S_x = f[\uparrow x]$, and define

$$e(U) = \{x \in Y : S_x \subseteq U\} = Y \setminus \downarrow f^{-1}(P \setminus U). \quad (6)$$

Cofinality makes every S_x nonempty, and $x \leq y$ implies $S_y \subseteq S_x$; hence $e(U)$ is an upset. Cofinality gives $e(0) = 0$; plainly $e(1) = 1$ and e preserves meets. It preserves implication as well. Namely, $x \notin e(U \rightarrow V)$ iff S_x contains $p \leq q$ with $q \in U \setminus V$. Choose $z \geq x$ in the domain with $f(z) = p$, and then use (PE2) to choose $w \geq z$ with $f(w) = q$. This gives $w \in e(U) \setminus e(V)$, so $x \notin e(U) \rightarrow e(V)$. Conversely, if $w \geq x$ belongs to $e(U) \setminus e(V)$, then some $q \in S_w \subseteq S_x$ lies in $U \setminus V$, so $x \notin e(U \rightarrow V)$.

If $(U, V) \in D$ and $S_x \subseteq U \cup V$, then $S_x \subseteq U$ or $S_x \subseteq V$. For a domain point this follows because S_x is principal. For a nondomain point, if S_x were contained in neither side,

choose witnesses in $S_x \setminus U$ and $S_x \setminus V$ and minimal points below them. Since U and V are upsets and $S_x \subseteq U \cup V$, these minimal points lie, respectively, in $V \setminus U$ and $U \setminus V$. Thus $\min S_x \in \mathcal{D}_{U,V}$, violating ZCDC. Hence $e(U \cup V) = e(U) \cup e(V)$. Finally, e is injective: if $p \in U \triangle V$, onto-ness gives a domain point x with $f(x) = p$, and then $x \in e(U)$ iff $p \in U$. Thus e is an embedding as in (ii). $\square$

Every generated cone of $M_{\mathbb{N}}$ is M_I for a nonempty $I \subseteq \mathbb{N}$. Therefore

$$\rho(\text{Up}(P), D) \notin \text{Skv} \iff \begin{array}{l} \text{for some nonempty finite or countably infinite } I, \text{ there is an} \\ \text{onto cofinal raw partial Esakia morphism } M_I \rightarrow P \\ \text{satisfying ZCDC}(\mathcal{D}_D). \end{array} \quad (7)$$

13.2 The quadrant tiling equivalence

Retain the finite pattern (P_W, D_W) constructed above and write $\rho_W = \rho(\text{Up}(P_W), D_W)$. A quadrant tiling is a map $\tau : \mathbb{N}^2 \rightarrow W$ satisfying the two Wang matching equations.

Theorem 13.2 (Infinite tiling equivalence). *For every finite nonempty Wang tile set W ,*

$$W \text{ tiles } \mathbb{N}^2 \iff \exists \emptyset \neq W' \subseteq W \quad \rho_{W'} \notin \text{Skv}.$$

Proof. For the right-to-left implication, use (7) to obtain $f : M_I \rightarrow P_{W'}$, where I is finite or countably infinite. Pair filling, the support-role lemma, the guard argument, seriality, and compatibility use only (PE1)–(PE3), cofinality, onto-ness, and ZCDC, so all remain valid for this raw map.

If I is finite, the earlier argument gives a finite torus tiling and hence a quadrant tiling. If I is infinite, the serial cuts make every vertex in each horizontal guard support G_{H_i} have an R_i -successor in $G_{H_{1-i}}$, and analogously for the vertical guard supports. Recursively choose alternating paths

$$h_0 \longrightarrow h_1 \longrightarrow \cdots, \quad v_0 \longrightarrow v_1 \longrightarrow \cdots.$$

Pair filling assigns a unique tile $\tau(a, b) \in W'$ to $\{h_a, v_b\}$. The horizontal and vertical compatibility cuts, applied to $\{h_a, h_{a+1}, v_b\}$ and $\{h_a, v_b, v_{b+1}\}$, give the Wang matching equations. Thus τ tiles $\mathbb{N}^2$.

Conversely, replace W by the nonempty range W' of a quadrant tiling, so that $\tau : \mathbb{N}^2 \rightarrow W'$. Take disjoint countably infinite blocks

$$H_i = \{h_i(a) : a \in \mathbb{N}\}, \quad V_j = \{v_j(b) : b \in \mathbb{N}\}, \quad \Omega = H_0 \sqcup H_1 \sqcup V_0 \sqcup V_1.$$

Use the earlier calibrated support families with

$$\begin{aligned} \{h_i(a), h_{1-i}(a+1)\} &\in \Sigma_{R_i}, \\ \{v_j(b), v_{1-j}(b+1)\} &\in \Sigma_{S_j}, \\ \{h_i(a), v_j(b)\} &\in \Sigma_{T_{ij, \tau(a,b)}}. \end{aligned}$$

All remaining cross-phase horizontal and vertical pairs receive the default labels R_{00} and S_{00} . Singleton supports, all same-role two-element supports, the whole-block guard supports, and the root support Ω are unchanged.

The successor matchings are disjoint, default pairs exist, and surjectivity of τ supplies every tile support in every phase. Thus calibration $\text{Act}(H) = \uparrow p$ for $H \in \Sigma_p$ is unchanged. Define

$$S \in \text{dom}(f), f(S) = p \iff \text{Act}(S) = \uparrow p.$$

The identity $f[\uparrow S] = \text{Act}(S)$ proves (PE1)–(PE3), onto-ness, and cofinality. Finiteness of the old source was used only for the omitted topological clauses.

The global ZCDC proof is likewise cardinal-independent. Pair cuts are filled by mixed two-element supports. At a serial cut, a nonoutput support contains a source vertex whose prescribed successor activates the output support. At a compatibility cut, either the relation support and cell support use distinct source vertices, activating the shield d_X , or they use the same source vertex, in which case the prescribed successor cell is activated and has a compatible tile label. The outside-point principle excludes a forbidden frontier in each case. Hence $f : M_\Omega \rightarrow P_{W'}$ satisfies ZCDC, and (7) yields $\rho_{W'} \notin \text{Skv}$. $\square$

Define effectively

$$\Phi_W = \bigwedge_{\emptyset \neq W' \subseteq W} \rho_{W'}.$$

Then

$$\Phi_W \in \text{Skv} \iff W \text{ does not tile } \mathbb{N}^2. \quad (8)$$

Theorem 13.3 (Skvortsov's logic). *The recursively axiomatizable logic Skv is undecidable, and*

$$\text{Skv} \subsetneq \text{ML}.$$

Proof. Quadrant tileability and plane tileability are equivalent. One direction is restriction. Conversely, a quadrant tiling supplies a valid pattern on a square of every finite size. The tree of all valid patterns on squares centred at the origin is finitely branching and has nodes at every height; König's lemma supplies a coherent branch and hence a tiling of $\mathbb{Z}^2$. The quadrant domino problem is therefore undecidable [2, 14], and the computable equivalence (8) proves that Skv is undecidable.

For strictness, choose an aperiodic Wang tile set W : it tiles the plane, but has no periodic tiling. The finite tiling equivalence gives $\Phi_W \in \text{ML}$, whereas (8) gives $\Phi_W \notin \text{Skv}$. Hence $\text{Skv} \subsetneq \text{ML}$. $\square$

Corollary 13.4 (A single canonical separator). *Let W be inclusion-minimal among the finite aperiodic Wang tile sets. Then*

$$\rho_W \in \text{ML} \setminus \text{Skv}.$$

Proof. Every plane tiling by W uses every tile: otherwise its proper range would itself tile the plane and could have no periodic tiling, contradicting minimality. The infinite completeness construction therefore refutes ρ_W on $M_{\mathbb{N}}$. The finite tiling equivalence makes ρ_W valid in ML, since W admits no periodic tiling. $\square$

The orientation in (8) is consistent with Skvortsov's recursive axiomatizability theorem: non-tileability is recursively enumerable by searching for an untileable finite square, while tileability is co-recursively enumerable by König's lemma. The varying instances W in the domino reduction prove undecidability; one fixed aperiodic tile set proves the strict

inclusion, but by itself is not an undecidability reduction. The separator in [corollary 13.4](#) is a bounded canonical formula with a proper selected join domain; it is not a Jankov formula.

Knudstorp's theorem itself concerns a binary associative modality over $(\mathcal{P}(\mathbb{N}), \cup)$ [14]; it does not entail [theorem 13.3](#) without the selected-join canonical pattern and [theorem 13.1](#). Indeed, that paper explicitly leaves the base modal and intuitionistic Skvortsov decision problems open immediately before proving undecidability of the richer conservative extension.

13.3 What this does not prove about the Jankov core

For each finite rooted P , the Jankov–de Jongh theorem gives

$$J(P) \notin \mathbf{ML} \iff P \text{ is a p-morphic image of some } M_n, \quad (9)$$

$$J(P) \notin \mathbf{Skv} \iff P \text{ is a p-morphic image of some } M_I \text{ with } \emptyset \neq I \subseteq \mathbb{N}. \quad (10)$$

Generated cones have been absorbed into n and I . Shehtman proves that these finite-base and arbitrary-base feasibility conditions agree for duplicate-free finite frames [21, Lemma 17 and Proposition 18]. Since $\mathbf{Skv} \subseteq \mathbf{ML}$, one always has the limited comparison

$$\mathbf{Jan}(\mathbf{Skv}) \subseteq Y_{\mathbf{ML}}.$$

Proposition 13.5 (Exact comparison with the Jankov core). *The equality $Y_{\mathbf{ML}} = \mathbf{Skv}$ holds if and only if both of the following conditions hold:*

- (a) *every finite rooted P which is a p-morphic image of some M_I , $\emptyset \neq I \subseteq \mathbb{N}$, is a p-morphic image of some finite M_n ;*
- (b) *$\mathbf{Skv} = \mathbf{Jan}(\mathbf{Skv})$, equivalently $\mathbf{Skv}$ is axiomatizable by Jankov formulas.*

Proof. Condition (a), together with (9)–(10), says exactly that $\mathbf{ML}$ and $\mathbf{Skv}$ contain the same Jankov formulas. Hence it is equivalent to $Y_{\mathbf{ML}} = \mathbf{Jan}(\mathbf{Skv})$. Adding (b) gives the displayed equality. Conversely, if $Y_{\mathbf{ML}} = \mathbf{Skv}$, then $\mathbf{Skv}$ is Jankov-axiomatizable and, for every finite rooted P ,

$$J(P) \in \mathbf{Skv} \iff J(P) \in Y_{\mathbf{ML}} \iff J(P) \in \mathbf{ML},$$

which is condition (a). □

Neither condition is established for arbitrary finite rooted frames. Shehtman's duplicate-free result verifies condition (a) only on that subclass; it says nothing about condition (b).

The tiling theorem does not fill this gap. Its formulas have a proper selected domain $D_W \subsetneq \text{Up}(P_W)^2$, and its countermodels are genuinely partial maps with many nondomain faces. Taking the full join domain turns the formula into an ordinary Jankov formula and requires a total p-morphism; the calibrated construction supplies no such map. Accordingly, the argument proves

$$\mathbf{Skv} \subsetneq \mathbf{ML} \quad \text{and} \quad \mathbf{Skv} \text{ is undecidable,}$$

but proves neither $Y_{\mathbf{ML}} = \mathbf{Skv}$ nor $Y_{\mathbf{ML}} \neq \mathbf{Skv}$. Either assertion requires a separate total-p-morphism or Jankov-axiomatizability theorem.

14 Simplicial and geometric form of the reduction

This section records two different, but compatible, senses in which the preceding construction is simplicial. The first is exact and topos-theoretic: Medvedev's logic is the universally quantified propositional theory of the ordinary presheaf topos of simplicial sets. The second is a face-filling description of the partial Esakia morphisms used in the reduction. The word “filling” is appropriate in the second description, but it must not be confused with the usual Kan horn-filling condition.

14.1 Finite Medvedev frames and standard simplices

Let Δ^m denote the standard abstract m -simplex, and let $F(\Delta^m)$ be its poset of nonempty faces, ordered by inclusion. To avoid conflict with the earlier notation for finite ground sets, write $[m]_\Delta = \{0, \dots, m\}$ for the object of the simplex category; it has $m + 1$ elements. Then

$$M_{m+1} = F(\Delta^m)^{\text{op}}.$$

It follows that

$$B_{m+1} = \text{Up}(M_{m+1}) \cong \text{Down}(F(\Delta^m)) \cong \text{Sub}_{\mathbf{sSet}}(\Delta[m]). \quad (*)$$

Here $\Delta[m] = y([m]_\Delta)$ is the representable simplicial set and the last algebra is its Heyting algebra of simplicial subsets. Equivalently, its elements are the possibly void abstract subcomplexes of the standard simplex, with the empty face suppressed. For subcomplexes K, L , Heyting implication is the largest subcomplex whose intersection with K is contained in L ; on nonempty vertex sets this is

$$K \Rightarrow L = \{S \subseteq [m]_\Delta : S \neq \emptyset \text{ and } (\forall \emptyset \neq T \subseteq S)(T \in K \implies T \in L)\}. \quad (**)$$

The order complex of M_{m+1} is, after forgetting the orientation of chains, the barycentric subdivision of Δ^m . Thus an order-preserving map between the posets under discussion induces a simplicial map between their order complexes. This standard bridge between finite posets, face posets, and polyhedra is used systematically in polyhedral semantics for intuitionistic logic; see [6, 1].

14.2 The propositional theory of the topos of simplicial sets

Write

$$\mathbf{sSet} = \widehat{\Delta} = \mathbf{Set}^{\Delta^{\text{op}}}.$$

For an elementary topos $\mathcal{E}$, define its *universally quantified propositional theory* by

$$\text{Th}_{\text{prop}}(\mathcal{E}) = \{\varphi(p_1, \dots, p_k) : \mathcal{E} \models \forall p_1, \dots, p_k : \Omega \varphi\}.$$

Equivalently, the interpretation $\llbracket \varphi \rrbracket : \Omega^k \rightarrow \Omega$ is the constant truth arrow. This is the precise meaning of “the propositional logic of a topos” used below.

Proposition 14.1 (Simplicial-set presentation of ML).

$$\text{Th}_{\text{prop}}(\mathbf{sSet}) = \bigcap_{m \geq 0} \text{Log Sub}_{\mathbf{sSet}}(\Delta[m]) = \text{ML}.$$

Proof. For a presheaf topos, $\Omega(c)$ is the Heyting algebra of sieves on c ; see [19, Chapter III]. We spell out the special case needed here. Associate with a sieve R on $[m]_\Delta$ the family

$$\kappa(R) = \{\text{im}(\alpha) : \alpha : [k]_\Delta \rightarrow [m]_\Delta \text{ belongs to } R\}.$$

This is a downset of the nonempty subsets of $[m]_\Delta$. Conversely, a downset K determines the sieve

$$R_K = \{\alpha : [k]_\Delta \rightarrow [m]_\Delta : \text{im}(\alpha) \in K\}.$$

These operations are inverse. Indeed, every monotone map α factors as the monotone surjection onto its image followed by the inclusion of that image; the surjection has a monotone section. Membership in a sieve is therefore determined exactly by the image of the map. Consequently

$$\Omega([m]_\Delta) \cong \text{Down}(\mathcal{P}^*([m]_\Delta)) \cong B_{m+1}$$

as Heyting algebras.

Equality of natural transformations is componentwise. Hence $\llbracket \varphi \rrbracket : \Omega^k \rightarrow \Omega$ is constantly true if and only if the operation defined by φ is constantly top on every finite Heyting algebra $\Omega([m]_\Delta)$. The claimed equality now follows from the definition of ML. This identification was also recorded explicitly by Jeřábek in [13]. $\square$

There is a necessary convention hidden in this statement. The global truth-value algebra $\text{Sub}_{\mathbf{sSet}}(1)$ is the two-element Boolean algebra. Looking only at subterminal objects would therefore yield classical propositional logic, not ML. Proposition 14.1 concerns propositions in arbitrary context, equivalently generalized elements of Ω , or the universal internal sequent with variables of type Ω . It concerns the ordinary one-topos $\mathbf{sSet}$, not the homotopy category of simplicial sets and not an ∞ -topos.

The main theorem therefore has the following exact geometric restatement.

Corollary 14.2 (Undecidable propositional theory of $\mathbf{sSet}$). *The set of intuitionistic propositional formulas valid, under all generalized truth-value assignments, in the topos $\mathbf{sSet}$ is co-recursively enumerable but not recursively enumerable. In particular, it is undecidable and has no recursively enumerable axiomatization.*

There is also a canonical-pattern version. Combining $(*)$ with Proposition 4.3, for every finite subdirectly irreducible Heyting algebra A and $D \subseteq A^2$,

$$\rho(A, D) \notin \text{ML} \iff \begin{array}{l} \text{for some } m, \text{ there is an injective map} \\ h : A \hookrightarrow \text{Sub}_{\mathbf{sSet}}(\Delta[m]) \end{array} \quad (\dagger)$$

preserving $0, 1, \wedge, \rightarrow$ and the joins selected by D . Thus it is undecidable whether a finite bounded implicative pattern with selected unions admits a faithful realization by subcomplexes of some finite standard simplex. Notice that the realization is allowed to use a simplex of unbounded dimension.

14.3 An exact selected face-filling problem

The recurrence of Proposition 5.1 can be turned into an intrinsic definition which no longer mentions a partial map. This gives the closest precise version of the “horn-filling” intuition behind the reduction.

Let P be a finite rooted poset. Write $\text{Ant}^+(P)$ for its nonempty antichains. An *antichain face profile* on Δ^m is a map

$$\lambda : \mathcal{P}^*([m]_\Delta) \longrightarrow \text{Ant}^+(P).$$

For a face S with $|S| > 1$, define its boundary profile by

$$\partial\lambda(S) = \min \bigcup_{i \in S} \uparrow \lambda(S \setminus \{i\}). \quad (\ddagger)$$

We call λ *D-admissible* when the following conditions hold.

(F1) For every vertex $\{i\}$, $\lambda(\{i\}) = \{p\}$ for some maximal $p \in P$.

(F2) For every $|S| > 1$, at least one of the following forms occurs:

$$\lambda(S) = \partial\lambda(S),$$

or, for some $p \in P$,

$$\lambda(S) = \{p\}, \quad \text{Cov}(p) = \partial\lambda(S).$$

The first form carries the boundary profile without adding a new least label. The second is a *cover-exact filling* of S by p .

(F3) Every $p \in P$ occurs: $\lambda(S) = \{p\}$ for some face S .

(F4) If $|\lambda(S)| > 1$, then $\lambda(S) \notin \mathcal{D}_D$.

The two alternatives in (F2) need not be disjoint as syntactic descriptions; the resulting profile is what matters.

Proposition 14.3 (Exact selected face-filling criterion). *For every finite rooted P and finite $D \subseteq \text{Up}(P)^2$,*

$$E(P, D) \iff \begin{array}{l} \text{some finite standard simplex carries a } D\text{-admissible} \\ \text{antichain face profile} \end{array}.$$

Proof. Given an $E(P, D)$ -witness $f : M_{m+1} \rightarrow P$, take $\lambda = \lambda_f$ from Proposition 5.1. Cofinality makes every profile nonempty. On a vertex, (PE2)–(PE3) force the unique label to be maximal. The recurrence gives (F2), onto-ness gives (F3), and ZCDC gives (F4).

Conversely, suppose λ is D -admissible. Put

$$\text{dom}(f) = \{S : \lambda(S) \text{ is a singleton}\}, \quad f(S) = p \text{ if } \lambda(S) = \{p\}.$$

An induction on $|S|$, using (F1)–(F2), gives the support-realization identity

$$\{f(T) : \emptyset \neq T \subseteq S, T \in \text{dom}(f)\} = \uparrow \lambda(S). \quad (\star)$$

For the carrying alternative this follows by taking the union over all facets. For a cover-exact filling use, in the finite poset P ,

$$\uparrow p = \{p\} \cup \uparrow \text{Cov}(p).$$

Identity $(\star)$ proves (PE1), the back condition (PE2), the exact-domain clause (PE3), and cofinality. The closedness clause (PE4) and the clopenness part of (PE5) follow from finiteness and discreteness; the set in (PE5) is an upset because it is the complement of a downset. Condition (F3) gives onto-ness, and (F4) is precisely ZCDC. $\square$

For fixed m, P, D , admissible profiles form a finite constraint problem. Proposition 14.3, the canonical-pattern theorem, and Corollary 11.1 yield the following.

Corollary 14.4 (Undecidable selected face filling). *The problem of deciding whether a given finite (P, D) admits a D -admissible profile on some finite standard simplex is recursively enumerable and undecidable. There is no computable upper bound, in terms of (P, D) , on the least possible dimension of a witnessing simplex.*

In the tiling instances, singleton labels provide the four kinds of coordinate vertices; binary faces carry horizontal, vertical, and tile labels; and compatibility is tested on faces generated by these supports. The pair-filling, serial, and compatibility members of D_W are exactly selected boundary profiles which are forbidden to remain nonprincipal. The guard and shield points ensure that resolving those profiles gives directed cycles and genuine incidence, so that the resulting finite filling system contains a toroidal Wang tiling. Conversely, such a tiling supplies the calibrated supports from which the profile is obtained. The subset device in Theorem 10.1 deals with tile symbols that do not occur in a chosen periodic tiling.

14.4 What is, and is not, a Kan condition

For comparison, the standard k -th horn is

$$\Lambda^k[n] = \bigcup_{i \neq k} d_i \Delta[n-1] \subseteq \Delta[n].$$

A Kan complex fills every map $\Lambda^k[n] \rightarrow X$, for every $0 \leq k \leq n$. A Kan fibration is defined by the corresponding right lifting property. A quasicategory (or inner Kan complex) is required to fill only the inner horns $0 < k < n$; an inner fibration is the relative version. In the Kan–Quillen setting, anodyne extensions form the weakly saturated class generated by all horn inclusions; inner anodyne maps are generated by the inner horn inclusions. We follow the standard terminology of [10, 18].

There is one literal lifting condition inside the present construction. For a partial Esakia morphism f , regard $f : \text{dom}(f) \rightarrow P$ as a map of posets. Its back condition (PE2) is the right lifting property in **Pos** against the initial-point inclusion

$$\{0\} \hookrightarrow [1].$$

After applying the nerve, this is the right lifting property against the oriented one-horn

$$\Lambda^0[1] \hookrightarrow \Delta[1].$$

Equivalently, by iterating (PE2), Nf has the right lifting property against the inclusion of the initial vertex $\Delta[0] \hookrightarrow \Delta[n]$ for every n : every finite target chain beginning at $f(S)$ has a lifted chain beginning at S . This is an order-theoretic open-map condition. It is only the one-dimensional outer-horn member of the standard left-fibration lifting family and does not make Nf a left or Kan fibration in general.

The remaining “fillings” are different in kind:

- Equation $(\ddagger)$ uses the *entire boundary* of a face (all of its facets), not a horn with one facet omitted.

- The boundary datum is an antichain of target labels, not in general a simplicial map $\partial\Delta[n] \rightarrow N(P)$.
- A filler is permitted only when that antichain is exactly $\text{Cov}(p)$ for some p ; it then labels the whole face by p . Arbitrary boundaries are not required to have fillers.
- ZCDC makes filling conditional on the selected semantic data D : a nonprincipal profile meeting both $U \setminus V$ and $V \setminus U$, while contained in $U \cup V$, may not be left unfilled.
- The decision problem asks whether there exists some finite ambient simplex, of unrestricted dimension, carrying an onto profile. Kan and inner-Kan conditions universally quantify over all horns in a fixed simplicial set or over a fixed map.

Accordingly, the theorem is best called an undecidability theorem for *selected cover-exact face filling*, or equivalently for finite selected partial- p -morphism patterns. It does not prove undecidability of Kan fibrancy, inner Kan fibrancy, or membership in an anodyne class. In particular, the finite family D is a family of selected joins (dually, forbidden antichain profiles); it is not a family of horn inclusions and is not an anodyne class.

Finally, this result is combinatorial rather than homotopy-invariant. Geometric realization sends the order complexes above to polyhedra, but it does not turn the Heyting implication $(**)$ into ordinary implication in the lattice of Euclidean open sets. The undecidability resides in face incidence, relative pseudocomplement, and selected unions in the one-topos **sSet**, not in the Kan–Quillen homotopy theory of spaces.

Remark 14.5 (The infinite frame). The full frame $(\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}, \supseteq)$ contains infinite subsets. It is therefore not the ordinary face poset of the countably infinite abstract simplex, whose faces are finite subsets. If geometric language is used for that frame, “power-set completion of the infinite simplex” is accurate; “the infinite simplex” without qualification is not.

15 Relation to the motivating search

The canonical-formula analysis produces a genuine simplification before the undecidability construction begins:

1. the arbitrary closed upset in the standard dual criterion can be replaced by a whole Medvedev frame, using [lemma 4.1](#);
2. structural completeness replaces the algebraic homomorphic image by a direct bounded implicative-semilattice embedding into B_n , using [proposition 4.3](#);
3. the resulting fixed- n search has the exact cover recurrence of [proposition 5.1](#).

What cannot be simplified away is the selected join domain D . The pair-filling, serial, and compatibility constraints in D_W are exactly what makes it possible to express a two-dimensional computation. The disjunction property does not constrain these local closed-domain conditions.

The construction was suggested in part by the use of powerset semilattices in recent tiling encodings for relevant and substructural logics, especially Knudstorp’s work [15]. A direct transplantation does not respect persistence and has the wrong semidecidability orientation

for the present problem. Passing to finite torus tilings, forcing cycles by guard faces, and adding the d_X -shields resolves those obstructions.

16 Verification notes

An independent adversarial audit was run after the finite and infinite reductions had been written. It separately checked the canonical-formula criterion, every use of the partial-morphism axioms in the finite gadget, the finite-type realization lemma in [theorem 13.1](#), the cardinality-independent support construction over $M_{\mathbb{N}}$, and the simplicial face recurrence. A separate second-pass audit checked the maximal-support quotient in [proposition 12.4](#) and again found that the unresolved step lies inside its fibers. No contradiction was found in those arguments. The same audit rejected the attempted inferences $Y_{\text{ML}} \neq \text{ML}$ and $Y_{\text{ML}} = \text{Skv}$: both require the additional total- p -morphism results isolated in [theorem 12.3](#) and [proposition 13.5](#). Those statements are accordingly not asserted in this note.

The proof separates the points at which an arbitrary witness could otherwise evade the intended local encoding.

- The support-role lemma makes no minimal-support assumption. It follows directly from the p -back condition.
- Pair filling proves that every active binary label on a relevant face is already visible on an actual two-face.
- The serial cuts use saturated intersections: every irrelevant nonroot label lies in $U \cap V$, so the only possible outside repairs are the intended successor and the root.
- On a compatibility triangle there is only one vertex of the source role, so neither d_X nor g_X can occur, and the missing fourth role excludes the root.
- In the converse construction, the support-realization identity $f[\uparrow S] = \text{Act}(S)$ verifies the exact-domain condition on every face, not merely on the displayed supports.
- The outside-point principle checks all larger forbidden antichains, rather than only their smallest examples.

Finite exhaustive checks were also performed on two constructed witnesses. For a three-tile cyclic 3×3 torus, the validator reported 31 target points, 12 coordinates, 4095 nonempty faces, 83 domain faces, 34 cuts, and zero failures. For a two-tile checkerboard 4×4 torus, it reported 27 target points, 16 coordinates, 65535 nonempty faces, 157 domain faces, 26 cuts, and zero failures. These computations are corroborative; the proof above treats arbitrary tile sets and arbitrary witness dimensions.

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